

CODEPTH TWO AND RELATED TOPICS

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Dedicated to Daniel Kastler on his eightieth birthday

ABSTRACT. A depth two extension $A|B$ is shown to be weak depth two over its double centralizer $V_A(V_A(B))$ if this is separable over B . We consider various examples and non-examples of depth one and two properties. Depth two and its relationship to direct and tensor product of algebras as well as cup product of relative Hochschild cochains is examined. Section 6 introduces a notion of codepth two coalgebra homomorphism $g : C \rightarrow D$, dual to a depth two algebra homomorphism. It is shown that the endomorphism ring of bicomodule endomorphisms $\text{End} {}^D C^D$ forms a right bialgebroid over the centralizer subalgebra $g^* : D^* \rightarrow C^*$ of the dual algebra C^* .

1. INTRODUCTION

From the quantum algebraic viewpoint, a depth two subalgebra is a notion that generalizes finite index normal Hopf subalgebra. Somewhat like a normal Hopf subalgebra is a Hopf-Galois extension, so does a depth two subalgebra admit a Galois theory of bialgebroid-valued coactions. The notion of depth two is also closely related to older definitions of normal subalgebra via representation theory, such as Rieffel's, where normality is defined by invariance of contracted maximal ideals with respect to the over-algebra. For example, a depth two extension has normal centralizer. Among Hopf subalgebras, the notion of normal subalgebra is the same as the notion of normal Hopf subalgebra. It follows that a depth two Hopf subalgebra has normal double centralizer subalgebra under certain circumstances [9]. In Section 2 of this paper we pursue a more general fact underlying this; that a depth two extension $A|B$ where the double centralizer $W = V_A(V_A(B))$ is a separable extension of B has *weak* depth two extension $A|W$. The definition of weak depth two, given in section 2, is modelled on Mewborn and McMahon's weakening of H-separability to "strong separability" in [14]. In section 3 we consider counterexamples and propositions for other closure and transitivity properties of depth two in a tower of algebras $A \supset C \supset B$. The aim is ultimately to extend Galois correspondences for special H-separable extensions to certain depth two algebra extensions. In section 4, we characterize a f.g. projective left Galois extensions with bialgebroid action in terms of its smash product being isomorphic to its right endomorphism ring. In section 5, we note that a one-sided depth two extension $A|B$ has relative Hochschild cochains with values in A generated w.r.t. the cup product by the 1-cochains.

The notion of bialgebroid dualizes to the notion of bicoalgebroid in [2]. It turns out that a weak Hopf algebra is both a bialgebroid and bicoalgebroid over its target subalgebra. To this author, it is an interesting continuation of this inquiry to dualize

depth two algebra homomorphism, define a codepth two coalgebra homomorphism and look for bicoalgebroids, or at least bialgebroids. One reason one might want to embark on this project is that a Hopf algebra homomorphism $K \rightarrow H$ is both an algebra and coalgebra homomorphism, so considering it as a codepth two coalgebra homomorphism is in principle just as interesting as considering it as a depth two algebra homomorphism; the latter having been done in terms of “depth two Hopf subalgebras,” the former then leading perhaps to interesting cases of “codepth two Hopf quotient algebras.” We carry out the beginnings of this project in section 6, where we define a left codepth two coalgebra homomorphism in terms of its associated cotensor product. The concept of codepth two is expressed in terms of coordinates called quasibases. It is then shown that the endomorphism algebra of the canonical bicomodule is a bialgebroid over the centralizer of the dual algebra homomorphism. We conclude with a discussion of how one might continue this investigation.

2. PRELIMINARIES AND WEAK DEPTH TWO

In this paper, algebras are unital associative over a commutative ground ring K and are themselves not necessarily commutative. An algebra extension $A|B$ is a unit-preserving algebra homomorphism $B \rightarrow A$, a proper extension if this mapping is monic. We focus mostly on the induced bimodule ${}_B A_B$ and mostly suppress the homomorphism. Unadorned tensors, hom-groups and endomorphism-groups between algebras are over the ground ring unless otherwise stated. In this section we denote $A^B = \{a \in A | \forall b \in B, ba = ab\}$, the centralizer $V_A(B)$ of B in A , although A^B should not be confused with the comodule notation in section 6, nor the invariant subring notation A^S where a bialgebroid S acts on A .

An algebra extension $A|B$ is *left depth two* (D2) if its tensor-square $A \otimes_B A$ as a natural B - A -bimodule is isomorphic to a direct summand of a finite direct sum of the natural B - A -bimodule A : for some positive integer N , we have

$$(1) \quad A \otimes_B A \oplus * \cong A^N$$

An extension $A|B$ is *right D2* if eq. (1) holds instead as natural A - B -bimodules. An algebra extension is of course D2 if it is both left D2 and right D2. For example, if $A|B$ is a faithfully flat algebra (so B is commutative, maps into the center of A and the module A_B is faithfully flat), then it is D2 if and only if A is f.g. projective over B , since $A \otimes_B A_A$ is f.g. projective.

Since condition (1) implies maps in two hom-groups satisfying $\sum_{i=1}^N g_i \circ f_i = \text{id}_{A \otimes_B A}$, where $g_i \in \text{Hom}({}_B A_A, {}_B A \otimes_B A_A) \cong (A \otimes_B A)^B$ (via $g \mapsto g(1)$) and

$$f_i \in \text{Hom}({}_B A \otimes_B A_A, {}_B A_A) \cong \text{End } {}_B A_B := S$$

via $f \mapsto (a \mapsto f(a \otimes_B 1))$, we obtain an equivalent condition for extension $A|B$ to be left D2: there is a positive integer N , $\beta_1, \dots, \beta_N \in S$ and $t_1, \dots, t_N \in (A \otimes_B A)^B$ (i.e., satisfying for each $i = 1, \dots, N$, $bt_i = t_i b$ for every $b \in B$) such that

$$(2) \quad \sum_{i=1}^N t_i \beta_i(x) y = x \otimes_B y$$

for all $x, y \in A$.

To see that the quasibases equation above is equivalent with the definition of left D2 in eq. (1) it remains to note the split epi of B - A -bimodules given by

$$(3) \quad A^N \longrightarrow A \otimes_B A, \quad (a_1, \dots, a_N) \longmapsto \sum_{i=1}^N t_i a_i,$$

with splitting map $x \otimes y \mapsto (\beta_1(x)y, \dots, \beta_N(x)y)$.

Eq. (2) is quite useful; for example, to show S finite projective as a left $V_A(B)$ -module (with module action given by $r \cdot \alpha = \lambda_r \circ \alpha$), apply $\alpha \in S$ to the first tensorands of the equation, set $y = 1$ and apply the multiplication mapping $\mu : A \otimes_B A \rightarrow A$ to obtain

$$(4) \quad \alpha(x) = \sum_i \alpha(t_i^1) t_i^2 \beta_i(x),$$

where we suppress a possible summation in $t_i \in A \otimes_B A$ using a Sweedler notation, $t_i = t_i^1 \otimes_B t_i^2$. But for each $i = 1, \dots, N$, we note that

$$T_i(\alpha) := \alpha(t_i^1) t_i^2 \in V_A(B) := R$$

defines a homomorphism $T_i \in \text{Hom}({}_R S, {}_R R)$, so that eq. (4) shows that $\{T_i\}$, $\{\beta_i\}$ are finite dual bases for ${}_R S$.

Similarly, an algebra extension $A|B$ is right D2 if there is a positive integer N , elements $\gamma_j \in \text{End}_B A_B$ and $u_j \in (A \otimes_B A)^B$ such that

$$(5) \quad x \otimes_B y = \sum_{j=1}^N x \gamma_j(y) u_j$$

for all $x, y \in A$. We call the elements $\gamma_j \in S$ and $u_j \in (A \otimes_B A)^B$ right D2 quasibases for the extension $A|B$. Fix this notation and the corresponding notation $\beta_i \in S$ and $t_i \in (A \otimes_B A)^B$ for left D2 quasibases throughout this paper.

In the paper [11] it is shown that a subgroup H of a finite group G has complex group algebras $\mathbb{C}H \subseteq \mathbb{C}G$ of depth two if and only if H is normal in G . From this fact we draw several examples to show that given an intermediate algebra $B \subset C \subset A$ there is in general no subalgebra pair $B \subset C$, $C \subset A$, or $B \subset A$, where being depth two will imply another subalgebra pair in these is depth two. For example, the trivial subgroup is normal in a finite group G containing a non-normal subgroup H , so that $A \supset B$ being D2 does not imply that $A \supset C$ is D2 (unlike separability).

If instead we ask if there may not be an exception to this rule if C bears some special relationship to B within A , a natural candidate comes to mind as the double centralizer of B in A . For we consider the “toy model” for D2 extensions, a type of “depth one” extension called H-separable extension. A result of Sugano and Hirata states that if $A|B$ is H-separable, then $A|V := V_A(V_A(B))$ is H-separable. The following example shows that this does not carry over verbatim to depth two extensions.

Example 2.1. Let $A = E(W_2)$, the exterior algebra of a vector space W_2 (over a field K of characteristic unequal to two) with basis $\{e_1, e_2\}$. Let B be the unit subalgebra, then its double centralizer V is the center of A , which is

$$V = K \cdot 1_A + K \cdot e_1 \wedge e_2.$$

The extension $A|B$ is D2 as is any finitely generated projective algebra. However, since $A|V$ is a split extension, if it were D2 the module A_V would be projective, hence free since V is a local algebra, which leads to a contradiction.

We need to add a hypothesis in order to obtain some result, such as $V|B$ is a separable extension. In preparation for the next theorem, we make a definition of a weakened notion of depth two, which is analogous to the weakening of H-separability in the notion of strong separability introduced in McMahon and Mewborn [14].

Definition 2.2. An algebra extension $A|B$ (with centralizer denoted by R) is a *weak left depth two* extension if the module ${}_R S$ is finitely generated (f.g.) projective and the left canonical B - A -homomorphism $\Psi : A \otimes_B A \rightarrow \text{Hom}({}_R S, {}_R A)$ defined below in eq. (6) is a split epi. The definition of a weak right D2 extension is defined oppositely. A weak depth two extension is weak right and left D2.

A left D2 extension $A|B$ is weak left D2 since ${}_R S$ f.g. projective and Ψ an isomorphism characterize left D2 extensions by [11, 2.1 (3)]. As an aside, the definition may be used to derive a split monomorphism $S \otimes_R S \hookrightarrow \text{Hom}({}_B A \otimes_B A_B, {}_B A_B)$ using [1, Prop. 20.11] (but with a different right R -module S than the one used below). The right R -module structure on S we will use for the rest of this paper is given by $\alpha \cdot r = \alpha(-)r$ for $r \in R = V_A(B)$, $\alpha \in \text{End } {}_B A_B = S$.

Theorem 2.3. *Let $W := V_A(V_A(B))$. If $A|B$ is a left (or right) D2 extension and $W|B$ is a separable extension, then $A|W$ is a weak left (or right) D2 extension.*

Proof. We note that $R = V_A(B)$ and $W = V_A(R)$ are in general each other's centralizers since also $V_A(W) = R$. Now consider the bimodule endomorphism algebra S' for the extension $A|W$ with base algebra $V_A(W) = R$. We claim that the natural inclusion $\iota : \text{End } {}_W A_W \hookrightarrow \text{End } {}_B A_B$ is a split R - R -monomorphism. Let $e = \bar{e} = e^1 \otimes e^2 \in W \otimes_B W$ be a separability element for $W|B$. Define a mapping $\eta : S \rightarrow S'$ by

$$\eta(\alpha) := \bar{\alpha} = e^1 \alpha (e^2 ? \bar{e}^1) \bar{e}^2$$

We note that $\eta(\lambda_r \rho_s \alpha) = \lambda_r \rho_s \eta(\alpha)$ for every $\alpha \in S$, $r, s \in R$ since elements in R and W commute. The mapping η is a splitting of ι since for every $\beta \in S'$ we have $\bar{\beta} = \beta$ as β is W -linear and $e^1 e^2 = 1$.

We will now show that $A|B$ left D2 implies $A|W$ is weak left D2. Since $A|B$ is left D2, the module ${}_R S$ is finite projective. Since ${}_R S'$ is a direct summand of ${}_R S$, it too is finite projective. It will then suffice to show that the mapping

$$(6) \quad \Psi : A \otimes_W A \rightarrow \text{Hom}({}_R S', {}_R A), \quad \Psi(x \otimes y)(\beta) := \beta(x)y$$

is a split W - A -epimorphism.

Define a splitting map

$$\text{Hom}({}_R S', {}_R A) \rightarrow A \otimes_W A, \quad G \longmapsto \sum_i e^1 t_i^1 \otimes_W t_i^2 e^2 G(\bar{\beta}_i).$$

Indeed, for each $\beta \in \text{End } {}_W A_W$,

$$\begin{aligned} \Psi\left(\sum_i e^1 t_i^1 \otimes_W t_i^2 e^2 G(\bar{\beta}_i)\right)(\beta) &= \sum_i \beta(e^1 t_i^1) t_i^2 e^2 G(\bar{\beta}_i) \\ &= \sum_i G(e^1 \beta(t_i^1) t_i^2 \beta_i (e^2 ? \bar{e}^1) \bar{e}^2) \\ &= G(e^1 \beta (e^2 ? \bar{e}^1) \bar{e}^2) = G(\beta) \end{aligned}$$

since $\beta(t_i^1)t_i^2 \in R$ for each i , G is left R -linear and $\sum_i \beta(t_i^1)t_i^2\beta_i(x) = \beta(x)$ for each $x \in A$ by eq. (4).

The proof that $W|B$ separable and $A|B$ right D2 implies that $A|W$ is weak right D2, is similar. \square

For example, the theorem is well-known for the complex group algebras corresponding to the subgroup situation where H normal in a finite group G implies that its centralizer $V_G(H)$ is normal in G . We provide a characterization for a weak left D2 extension. Note that the B - A -subbimodule U below coincides with the reject of A in $A \otimes_B A$ [1, p. 109].

Proposition 2.4. *An algebra extension $A|B$ is weak left D2 if and only if as B - A -bimodules, the three conditions below are satisfied:*

- (1) $A \otimes_B A = U \oplus L$, where
- (2) $\text{Hom}(U, A) = \{0\}$, and
- (3) $L \oplus * \cong A^N$ for some positive integer N .

Proof. (\Rightarrow) Since ${}_R S$ is f.g. projective, there is a positive integer N such that ${}_R S \oplus * \cong {}_R R^N$. Then $\text{Hom}({}_R S, {}_R A) \oplus * \cong A^N$ as B - A -bimodules. But $\Psi : A \otimes_B A \rightarrow \text{Hom}({}_R S, {}_R A)$ defined by $\Psi(x \otimes y)(\alpha) = \alpha(x)y$ is a split epi. Let L be the image in $A \otimes_B A$ of $\text{Hom}({}_R S, {}_R A)$ under a split monic, therefore satisfying condition (3). Let $U = \ker \Psi$. Then $A \otimes_B A = U \oplus L$. We next show that $\text{Hom}_{B-A}(U, A) = 0$ where the unlabelled homomorphism groups are w.r.t. B - A -bimodules:

$$S \xrightarrow{\cong} \text{Hom}(A \otimes_B A, A) \cong \text{Hom}(\text{Hom}({}_R S, {}_R A), A) \oplus \text{Hom}(U, A) \cong S \oplus \text{Hom}(U, A)$$

since ${}_R S$ f.g. projective implies $\text{Hom}(\text{Hom}({}_R S, {}_R A), A) \cong \text{Hom}(A, A) \otimes_R S \cong S$ by a standard isomorphism (e.g., [14, 3.4]) and the fact that $\text{End}({}_B A_A) \cong R$ via $f \mapsto f(1)$. The first arrow above is given by $\alpha \mapsto (x \otimes_B y \mapsto \alpha(x)y)$ with inverse $F \mapsto F(? \otimes 1_A)$. It is not hard to check that the composite isomorphism is the identity on the direct summand S , whence $\text{Hom}(U, A) = 0$.

(\Leftarrow) Since $S \cong \text{Hom}(A \otimes_B A, A)$, it follows from substitution of condition (1) and applying condition (2) that $S \cong \text{Hom}({}_B L_A, {}_B A_A)$. From condition (3) and a derivation as in that of eq. (2), we arrive at elements $2N$ elements $v_j = v_j^1 \otimes v_j^2 \in L^B \subseteq (A \otimes_B A)^B$, $\delta_j \in S$ such that

$$(7) \quad \ell = \sum_{j=1}^N v_j^1 \otimes_B v_j^2 \delta_j (\ell^1) \ell^2$$

for all $\ell = \ell^1 \otimes \ell^2 \in L \subseteq A \otimes_B A$. Again from condition (3), $\text{Hom}(L, A) \oplus * \cong \text{End}({}_B A_A)^N$, whence ${}_R S \oplus * \cong {}_R R^N$ and S is left f.g. projective R -module.

The mapping $\Psi : A \otimes_B A \rightarrow \text{Hom}({}_R S, {}_R A)$ is split by the B - A -bimodule homomorphism

$$\sigma : \text{Hom}({}_R S, {}_R A) \rightarrow A \otimes_B A, \quad \sigma(G) := \sum_{i=1}^N v_j^1 \otimes_B v_j^2 G(\delta_j)$$

since $\sum_j \alpha(v_j^1)v_j^2 G(\delta_j) = G(\sum_j \alpha(v_j^1)v_j^2 \delta_j) = G(\alpha)$ (for all $\alpha \in S \cong \text{Hom}(L, A)$), which follows from eq. (7). \square

From the proof it is clear that another characterization of weak left D2 extension $A|B$ is that its tensor-square has a direct sum decomposition as in conditions (1)

and (2), where all elements of L satisfy a left quasi-bases equation (7). The extent to which a weak depth two extension has a Galois theory might be an interesting problem.

3. FURTHER CLOSURE PROPERTIES OF DEPTH TWO WITH COUNTEREXAMPLES

Unlike separable extensions and Frobenius extensions, depth two is not a transitive property. If G is a finite group with normal subgroup N having a normal subgroup K where $K \not\triangleleft G$, then the corresponding complex group algebras $A = \mathbb{C}G$, $B = \mathbb{C}K$, and $C = \mathbb{C}N$ satisfy $A \supset C \supset B$ with $A|C$ and $C|B$ both D2 but $A|B$ not D2. However, normality of subgroups $G \geq N \geq K$ satisfies $K \triangleleft G$ normal $\Rightarrow K \triangleleft N$. Thus it may come as a surprise that $A|C$ and $A|B$ D2 $\nRightarrow C|B$ D2, which may be seen from the example $A = M_2(\mathbb{C})$, $B = \mathbb{C} \times \mathbb{C}$ and $C = T_2(\mathbb{C})$, the triangular and full 2×2 matrix algebras and the algebra of diagonal matrices, as shown in the next proposition.

Proposition 3.1. *The matrix algebras $A = M_n(k)$, $B = \text{Diag}_n(k)$ and $C = T_n(k)$ over any field k satisfy: $A|B$ and $A|C$ are H-separable (and therefore D2), but $C|B$ is not D2.*

Proof. An algebra extension $A|C$ is H-separable iff $1 \otimes_B 1$ may be expressed as a sum of products of elements in the centralizer $V_A(C)$ and (Casimir) elements in $(A \otimes_C A)^A$. Recall that for any fixed $j = 1, \dots, n$, $\sum_{i=1}^n e_{ij} \otimes_k e_{ji}$ is a Casimir element in $(A \otimes_k A)^A$. But note that $1 \otimes_C 1$ reduces to a canonical image of this Casimir element, since the matrix units $e_{ii}, e_{1i} \in C$ for $1 \leq i \leq n$ yields

$$1 \otimes_C 1 = \sum_{i=1}^n e_{ii} \otimes_C e_{ii} = \sum_{i=1}^n e_{i1} \otimes_C e_{1i}.$$

For a similar reason $A|B$ is H-separable, since the centralizer $V_A(B) = B$, and

$$1 \otimes_B 1 = \sum_{i=1}^n e_{ii} \otimes_B e_{ii} = \sum_{i=1}^n e_{ii} \left(\sum_{j=1}^n e_{ji} \otimes_B e_{ij} \right).$$

Now, if an extension $C|B$ is D2, then its centralizer $R := V_C(B)$ is a normal subalgebra in C : i.e., for each two-sided ideal I in C , we have the C -invariance of the ideal contracted to R , $(R \cap I)C = C(R \cap I)$ [9, Prop. 4.2]. But $R = B$ in our example and C has the two-sided ideal $I = \sum_{i=1}^n k e_{1i}$ where $C(I \cap B) \neq (I \cap B)C$. Hence, $C|B$ is not D2. \square

It is similarly shown that $C|B$ is not one-sided D2. There is a certain transitivity of the depth two property when it follows an H-separable extension.

Proposition 3.2. *If the algebra extension $A|C$ is right (or left) D2 and the extension $C|B$ is H-separable, then $A|B$ is right (or left) D2.*

Proof. If $C|B$ is H-separable, we have

$${}_C C \otimes_B C_C \oplus * \cong {}_C C^N {}_C$$

for some positive integer N . Apply the functor ${}_C \mathcal{M}_C \rightarrow {}_A \mathcal{M}_C$ given by ${}_A A \otimes_C ? \otimes_C A_C$, to obtain

$$(8) \quad {}_A A \otimes_B A_C \oplus * \cong \bigoplus^N {}_A A \otimes_C A_C.$$

If $A|C$ is right D2, we have ${}_A A \otimes_C A_C \oplus * \cong {}_A A^M_C$ for some positive integer M . It follows from this applied to eq. (8), then restricting from right C -modules to B -modules that

$${}_A A \otimes_B A_B \oplus * \cong {}_A A^{NM}_B,$$

which is the condition that $A|B$ is right D2. It is similarly proven that a left D2 following an H-separable extension is altogether left D2. \square

There has been a question of whether a right or left progenerator H-separable extension $A|B$ is split (i.e., has a B - B -bimodule projection $A \rightarrow B$), whence Frobenius: an affirmative answer implies some generalizations of results of Noether-Brauer-Artin on simple algebras [17]. Unfortunately, the next example, derived from the endomorphism ring theorem for D2 extensions in [10], of a one-sided free H-separable non-Frobenius extension rules out this possibility.

Example 3.3. Let K be a field and B the 3-dimensional algebra of upper triangular 2×2 -matrices, which is not self-injective. Since $B|K1$ is trivially D2, the endomorphism algebra $A := \text{End } B_K \cong M_3(K)$ is a left D2 extension of $\lambda(B)$, which w.r.t. the ordered basis $\langle e_{11}, e_{12}, e_{22} \rangle$ $\lambda(B)$ is the subalgebra of matrices

$$[x, y, z] := \begin{pmatrix} x & 0 & 0 \\ 0 & x & y \\ 0 & 0 & z \end{pmatrix}$$

Now the centralizer R is the 3-dimensional algebra spanned by matrix units e_{11} , e_{21} , $e_{22} + e_{33}$. The module ${}_B A$ is free with left B -module isomorphism $A \rightarrow B^3$ given by “separating out the columns”

$$(a_{ij}) \mapsto ([a_{11}, a_{21}, a_{31}], [a_{12}, a_{22}, a_{32}], [a_{13}, a_{23}, a_{33}])$$

whence $A \otimes_B A$ and $\text{Hom}(R_K, A_K)$ are both 27-dimensional. The A - A -homomorphism $A \otimes_B A \rightarrow \text{Hom}(R_K, A_K)$ given by $a \otimes c \mapsto (r \mapsto arc)$ is easily computed to be surjective, therefore an isomorphism, whence ${}_A A \otimes_B A_A \cong {}_A A_A^3$, which shows $A|B$ is H-separable (and D2). The extension $A|B$ is not Frobenius since B is not a Frobenius algebra; therefore $A|B$ is not split. (Alternatively, if there is a B -linear projection $E : A \rightarrow B$, we note $E(e_{32}) = 0$, so $e_{33} = e_{32}e_{23} \in \ker E$, a contradiction.) By applying the matrix transpose, the results of this example may be transposed to a right-sided version.

Next we deal with elementary ways to generate new depth two extensions from old ones. In our first case, we note an extension is depth two if and only if all its components in a finite direct product are D2.

Proposition 3.4. *Suppose $B_k \rightarrow A_k$ is an algebra homomorphism ι_k for each $k = 1, \dots, n$. Let $B = B_1 \times \dots \times B_n$, $A = A_1 \times \dots \times A_n$ and $B \rightarrow A$ be induced by $\iota = \times_{i=1}^n \iota_i$. Then $A|B$ is D2 if and only if $A_k|B_k$ is D2 for each $k = 1, \dots, n$.*

Proof. Let $p_i : A \rightarrow A_i$ and $\sigma_i : A_i \rightarrow A$ be the canonical algebra morphisms satisfying $\sum_{i=1}^n \sigma_i \circ p_i = \text{id}_A$, $p_i \circ \sigma_j = \text{id}_{A_i} \delta_{ij}$. Let $e_i = \sigma_i \circ p_i(1_A)$ be the canonical orthogonal idempotents. Similarly, let $\pi_i : B \rightarrow B_i$ and $\eta_i : B_i \rightarrow B$ be the corresponding canonical mappings and orthogonal idempotents $f_i = \eta_i \circ \pi_i(1_B)$. These satisfy commutative squares corresponding to $p_i \circ \iota = \iota_i \circ \pi_i$, $\sigma_i \circ \iota_i = \iota \circ \eta_i$ and $\iota(f_i) = e_i$ for each $i = 1, \dots, n$.

Note that $p_i \otimes p_i$ induces a B - B -bimodule epimorphism from $A \otimes_B A \rightarrow A_i \otimes_{B_i} A_i$ split by $\sigma_i \otimes \sigma_i$. Since $\sigma_i(A_i) \otimes_B \sigma_j(A_j) = 0$ where $i \neq j$ by using e_i , we see that $(A \otimes_B A)^B \cong \bigoplus_{i=1}^n (A_i \otimes_{B_i} A_i)^{B_i}$.

Similarly, $\text{End}_B A_B \cong \times_{i=1}^n \text{End}_{B_i}(A_i)_{B_i}$ via $\alpha \mapsto (p_i \circ \alpha \circ \sigma_i)_{i=1}^n$. Also, the centralizer $V_A(B) = \times_{i=1}^n V_{A_i}(B_i)$.

The proof is now completed by projecting eq. (2) via $p_i \otimes p_i$ onto n left D2 quasibases equations for $A_i | B_i$, and conversely gluing together n left D2 quasibases equations into one for $A | B$. A similar argument using the right D2 quasibases eq. (5) shows $A | B$ is right D2 \Leftrightarrow each $A_i | B_i$ is right D2. \square

The proof shows that the R -bialgebroids S and T for $A | B$ are not surprisingly direct products of R_i -bialgebroids $S_i = \text{End}_{B_i}(A_i)_{B_i}$ and $T_i = (A_i \otimes_{B_i} A_i)^{B_i}$, respectively, where $R_i = V_{A_i}(B_i)$. We retain this notation in dealing with the tensor product of algebras next. Tensor product of finitely many D2 algebra extensions is D2, but the converse is more demanding and requires several hypotheses. Suppose then that all the algebras in the proposition below are finite dimensional algebras over a field K and a reminder that unadorned tensors are over K .

Proposition 3.5. *Let $B_k \rightarrow A_k$ be algebra homomorphisms ι_k for each $k = 1, \dots, n$ where each B_k is a separable algebra. Let $B = B_1 \otimes \dots \otimes B_n$, $A = A_1 \otimes \dots \otimes A_n$ and $B \rightarrow A$ be induced by $\iota = \bigotimes_{i=1}^n \iota_i$. Then $A | B$ is D2 if and only if $A_k | B_k$ is D2 for each $k = 1, \dots, n$.*

Proof. (\Leftarrow) Here no hypotheses are required beyond the objects defined. We note the simple rearrangement mapping $\bigotimes_{i=1}^n T_i \rightarrow T$, as well as $\bigotimes_{i=1}^n S_i \rightarrow S$ given by sending $\alpha_1 \otimes \dots \otimes \alpha_n$ to itself (for $\alpha_i \in S_i$). Let $u_{ij} \in T_i$ and $\gamma_i \in S_i$ be given for each i satisfying the right D2 quasibases eq.

$$a_i \otimes_{B_i} a_i' = \sum_{j=1}^n a_i \gamma_{ij}(a_i') u_{ij}^1 \otimes_{B_i} u_{ij}^2,$$

for $a_i, a_i' \in A_i$. Then $\gamma_j = \bigotimes_{i=1}^n \gamma_{ij} \in S$ and

$$u_j = (u_{1j}^1 \otimes_K \dots \otimes_K u_{nj}^1) \otimes_B (u_{1j}^2 \otimes_K \dots \otimes_K u_{nj}^2) \in T$$

as well as $a = a_1 \otimes_K \dots \otimes_K a_n$, $a' = a_1' \otimes_K \dots \otimes_K a_n'$ satisfy

$$a \otimes_B a' = \sum_j a \gamma_j(a') u_j^1 \otimes_B u_j^2.$$

A similar argument shows $A | B$ is also left D2.

(\Rightarrow) We need a lemma [5, 2.4] stating that for A and B two K -algebras with finite projective modules ${}_A M$ and ${}_B N$ and two others ${}_A M'$ and ${}_B N'$, the natural mapping

$$(9) \quad \text{Hom}({}_A M, {}_A M') \otimes_K \text{Hom}({}_B N, {}_B N') \xrightarrow{\cong} \text{Hom}_{A \otimes B}(M \otimes N, M' \otimes N')$$

is an isomorphism (with inverse $F \mapsto \sum_{i,j} f_i \otimes g_j F(m_i \otimes n_j)$ where m_i, f_i are dual bases for ${}_A M$ and n_j, g_j are dual bases for ${}_B N$). This fact may be extended by induction to any number of tensor factors.

Since each $B_i^e := B_i \otimes B_i^{\text{op}}$ is semisimple, the B_i^e -modules A_i are finite projective, whence

$$S_1 \otimes_K \dots \otimes_K S_n \cong S,$$

since $B^e \cong B_1^e \otimes \dots \otimes B_n^e$.

Next recall that for any algebra B and every B - B -bimodule V the subgroup of B -central elements, $V^B \cong \text{Hom}_{B^e}(B, V)$ in a functorial way, which is right exact if B is B^e -projective, i.e., B is a separable algebra [5]. It then follows from $A \otimes_B A \cong \otimes_{i=1}^n A_i \otimes_{B_i} A_i$ that

$$T \cong \text{Hom}_{B^e}(B, A \otimes_B A) \cong \otimes_{i=1}^n \text{Hom}_{B_i^e}(B_i, A_i \otimes_{B_i} A_i) = T_1 \otimes \cdots \otimes T_n.$$

Similarly, $R = A^B = R_1 \otimes \cdots \otimes R_n$.

We will use the characterization that $A|B$ is left D2 if T_R is finite projective and $T \otimes_R A \xrightarrow{\cong} A \otimes_B A$ via the map $\beta(t \otimes_R a) := t^1 \otimes_B t^2 a$ [11, 2.1(4)]. Under the decompositions above, β decomposes into corresponding mappings

$$\otimes_{i=1}^n \beta_i : \otimes_{i=1}^n T_i \otimes_{R_i} A_i \xrightarrow{\cong} \otimes_{i=1}^n A_i \otimes_{B_i} A_i.$$

By finite dimensionality one shows that each β_i is injective and surjective. Finally, T_R is finite projective, so T_{R_i} is finite projective, whence the summand T_i is finite projective as a right R_i -module. Thus $A_i|B_i$ is left D2 for each $i = 1, \dots, n$, and by a similar argument, it is right D2. \square

From the proof we note that the R -bialgebroids S and T for a (one-sided or two-sided) D2 extension $A|B$ once again decompose, this time into a tensor product of R_i -bialgebroids S_i and T_i (with antipode if the characteristic of K is zero [10, 3.6]. Note that any algebra is a D2 extension of itself.

Corollary 3.6. *Let B be a separable algebra. Then $A|B$ is D2 $\Leftrightarrow M_n(A)|M_n(B)$ is D2.*

4. A CHARACTERIZATION OF GALOIS EXTENSION

Tachikawa [18] studies the double centralizer condition for modules and the property of being balanced for rings. We give a related result — that a subalgebra satisfying the double centralizer (or bicommutant) condition has a balanced right or left regular representation on the over-algebra.

Lemma 4.1. *If $B \subseteq A$ is a subalgebra, then*

$$B \subseteq A^S := \{x \in A : \forall \alpha \in \text{End}_B A_B, \alpha(x) = \alpha(1_A)x\} \subseteq V_A(V_A(B)).$$

If B satisfies the double centralizer condition, $V_A(V_A(B)) = A$, or equals the invariant subalgebra $A^S = B$, then the natural modules A_B and ${}_B A$ are balanced.

Proof. Again let S denote $\text{End}_B A_B$. Clearly $B \subseteq A^S$. Assume $x \in A^S$. Then for each $r \in V_A(B)$ we have $\rho_r \in S$, so $\rho_r(x) = \rho_r(1)x$, i.e. $xr = rx$, so $x \in V_A(V_A(B))$. Hence, $B \subseteq A^S \subseteq V_A(V_A(B))$.

Suppose B satisfies the double centralizer condition in A . Then $A^S = B$. To see that A_B is balanced, let E denote $\text{End}_B A_B$ and consider $f \in \text{End}_E A$. Then $\forall a \in A$,

$$f(a) = f(\lambda_a(1)) = \lambda_a(f(1)) = af(1),$$

thus $x := f(1)$ satisfies $f = \rho_x$. It suffices to show that $x \in B$. Let $\alpha \in S$, then

$$\alpha(x) = f(\alpha(1)) = \lambda_{\alpha(1)}f(1) = \alpha(1)x$$

whence $x \in A^S = B$. The proof that ${}_B A$ is balanced is similar. \square

Recall that an algebra extension $A|B$ is a \mathcal{G} -Galois extension [16] if \mathcal{G} is a group of automorphisms of the algebra A fixing each element in B such that

- (1) the natural module A_B is finitely generated and projective;

- (2) the mapping $j : A \rtimes G \rightarrow \text{End } A_B$ given by $j(a\#\sigma)(x) = a\sigma(x)$ (for each $a, x \in A, \sigma \in G$) is an isomorphism;
- (3) the set of invariants $A^G = \{x \in A : \sigma(x) = x, \forall \sigma \in G\}$ is equal to B : $A^G = B$.

Now consider left or right Galois extensions for bialgebroids and their characterization as left or right depth two and balanced extensions [8]. Next we give a characterization for finite projective Galois extensions which is very similar to the one above for group-Galois extensions:

Theorem 4.2. *Suppose $A|B$ is an algebra extension with centralizer denoted by $R = V_A(B)$ and A_B finite projective. Then $A|B$ is a left Galois extension iff ${}_R S$ is finite projective, $j : A \otimes_R S \rightarrow \text{End } A_B$ given by $j(a \otimes \alpha)(x) = a\alpha(x)$ is an isomorphism, and $A^S = B$.*

Proof. (\Rightarrow) From [8, Theorem 2.1], $A|B$ is left D2 and balanced, and from [12, 3.10, 4.1], j is an isomorphism and $A^S = B$. That ${}_R S$ is finite projective is noted above after eq. (4).

(\Leftarrow) If ${}_R S \oplus * \cong {}_R R^N$, we tensor by $A \otimes_R -$ and apply the A - B -isomorphism j to obtain $\text{End } A_B \oplus * \cong A^N$ as natural A - B -bimodules. Since A_B is finite projective, we may apply [9, 3.8] to see that $A|B$ is left D2. Since $A^S = B$, Lemma 4.1 informs us that A is balanced over B . Thus, $A|B$ is a left Galois extension by [8, Theorem 2.1]. \square

5. DEPTH TWO AND CUP PRODUCT IN SIMPLICIAL HOCHSCHILD COHOMOLOGY

Let $A|B$ be an extension of K -algebras. We briefly recall the B -relative Hochschild cohomology of A with coefficients in A (for coefficients in a bimodule, see the source [7]). The zero'th cochain group $C^0(A, B; A) = A^B = R$, while the n 'th cochain group $C^n(A, B; A) = \text{Hom}_{B-B}(A \otimes_B \cdots \otimes_B A, A)$ (n times A in the domain). In particular, $C^1(A, B; A) = S = \text{End } {}_B A_B$. The coboundary $\delta_n : C^n(A, B; A) \rightarrow C^{n+1}(A, B; A)$ is given by

$$\begin{aligned} (\delta_n f)(a_1 \otimes \cdots \otimes a_{n+1}) &= a_1 f(a_2 \otimes \cdots \otimes a_{n+1}) + (-1)^{n+1} f(a_1 \otimes \cdots \otimes a_n) a_{n+1} \\ (10) \quad &+ \sum_{i=1}^n (-1)^i f(a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+1}) \end{aligned}$$

which satisfies $\delta_{n+1} \circ \delta_n = 0$ for each positive integer n . Its cohomology is denoted by $H^n(A, B; A) = \ker \delta_n / \text{Im } \delta_{n-1}$, and might be referred to as a simplicial Hochschild cohomology, since this cohomology is isomorphic to simplicial cohomology if A is the poset algebra of a simplicial complex [6].

The cup product $\cup : C^m(A, B; A) \otimes_K C^n(A, B; A) \rightarrow C^{n+m}(A, B; A)$ makes use of the multiplicative structure on A and is given by

$$(11) \quad (f \cup g)(a_1 \otimes \cdots \otimes a_{n+m}) = f(a_1 \otimes \cdots \otimes a_m) g(a_{m+1} \otimes \cdots \otimes a_{n+m})$$

which satisfies the equation $\delta_{n+m}(f \cup g) = (\delta_m f) \cup g + (-1)^m f \cup \delta_n g$ [6]. Cup product therefore passes to a product on the cohomology. We note that $(C^*(A, B; A), \cup, +, \delta)$ is a differential graded algebra (perhaps negatively graded according to the convention used) which we denote by $D(A, B)$.

Theorem 5.1. *Suppose $A|B$ is a right or left D2 algebra extension. Then $D(A, B)$ is generated as an algebra by its degree one elements and is isomorphic to the tensor algebra on $C^1(A, B; A)$ over $C^0(A, B; A)$.*

Proof. The idea of the proof is to generalize the isomorphism

$$S \otimes_R S \xrightarrow{\cong} \text{Hom}({}_B A \otimes_B A_B, {}_B A_B)$$

via $\alpha_1 \otimes_R \alpha_2 \mapsto \alpha_1 \cup \alpha_2$ [12, 3.11] in the notation for $S = C^1(A, B; A)$ and $R = C^0(A, B; A)$ above. This shows that any 2-cochain is the cup product of 1-cochains. Similarly, any n -cochain is the cup product of 1-cochains, since

$$(12) \quad S \otimes_R \cdots \otimes_R S \xrightarrow{\cong} \text{Hom}_{B-B}(A \otimes_B \cdots \otimes_B A, A).$$

via $\alpha_1 \otimes \cdots \otimes \alpha_n \mapsto \alpha_1 \cup \cdots \cup \alpha_n$. We prove this by induction on n , the statement holding in $n = 1, 2$. Suppose it holds for $n < m$. Note that $S \otimes_R C^{m-1}(A, B; A) \xrightarrow{\cong} C^m(A, B; A)$ via $\alpha \otimes g \mapsto \alpha \cup g$ since an inverse is clearly given by $f \mapsto \sum_j \gamma_j \otimes_R u_j^1 f(u_j^2 \otimes_B \cdots \otimes_B -)$ in terms of right D2 quasibases. By the induction hypothesis $C^{m-1}(A, B; A) \cong S \otimes_R \cdots \otimes_R S$ ($m-1$ times S) via the cup product, so the proof is complete. It follows that there is an isomorphism of algebras,

$$(13) \quad T^*({}_R S_R) \xrightarrow{\cong} D(A, B)$$

via the cup product mapping above. \square

The inverse mapping for $S \otimes_R S \otimes_R S \cong \text{Hom}({}_B A \otimes_B A \otimes_B A_B, {}_B A_B)$ implied by the proof has a different expression than that in [8, (46)]: it comes out here as $g \mapsto \sum_{i,j} \gamma_j \otimes_R \gamma_i \otimes_R u_i^1 u_j^1 g(u_j^2 \otimes_B u_i^2 \otimes_B -)$ for $g \in C^3(A, B; A)$.

6. CODEPTH TWO COALGEBRA HOMOMORPHISMS AND THEIR BIALGEBROIDS

In this section, we dualize the notion of depth two for algebra homomorphisms to obtain a notion of codepth two for coalgebra homomorphisms. We obtain workable codepth two quasibases via simplifications of certain hom-groups of comodule homomorphism. We then establish a right bialgebroid structure on the bicomodule endomorphisms, where the base algebra is the centralizer of the dual algebra homomorphism.

Let $g : C \rightarrow D$ be a homomorphism of coalgebras over a field (alternatively, coalgebras which are flat over a base ring K with K -duals that separate points). Then C has an induced D - D -bicomodule structure given by left coaction

$$\rho^L : C \rightarrow D \otimes C, \quad \rho^L(c) = c_{(-1)} \otimes c_{(0)} := g(c_{(1)}) \otimes c_{(2)},$$

and by right coaction

$$\rho^R : C \rightarrow C \otimes D, \quad \rho^R(c) = c_{(0)} \otimes c_{(1)} := c_{(1)} \otimes g(c_{(2)}).$$

In a similar way, any C -comodule becomes a D -comodule via the homomorphism g , the functor of corestriction [3, 11.9]. We denote a right D -comodule M by M^D for example. This should not be confused with our notation A^B for the centralizer of a subring B in a ring A , which we need to work with simultaneously below. (Unadorned tensors between modules are over K , we use a generalized Sweedler notation, the identity is sometimes denoted by its object, and basic terminology such as coalgebra homomorphism, comodule or bicomodule is defined in the standard way such as in [3].)

Recall that the cotensor product

$$C \square_D C = \{c \otimes c' \in C \otimes C \mid c_{(1)} \otimes g(c_{(2)}) \otimes c' = c \otimes g(c'_{(1)}) \otimes c'_{(2)}\},$$

where we suppress a possible summation $c \otimes c' = \sum_i c_i \otimes c'_i$. For example, if $g = \varepsilon : C \rightarrow K$ the counit on C , $C \square_D C = C \otimes C$.

Note that $C \square_D C$ is a natural C - C -bicomodule via the coproduct Δ on C applied as $\Delta \otimes C$ for the left coaction and $C \otimes \Delta$ for the right coaction [3, 11.3]. Then $\underline{\Delta} : C \rightarrow C \square_D C$ induced by Δ (where $\underline{\Delta}(c) := c_{(1)} \otimes c_{(2)}$) is a C - C -bicomodule monomorphism. As D - C -bicomodule it is split by $c \otimes c' \mapsto \varepsilon(c)c'$, and as a C - D -bicomodule $\underline{\Delta}$ is split by $c \otimes c' \mapsto c\varepsilon(c')$ for $c \otimes c' \in C \square_D C$. (Since $c_{(1)} \otimes g(c_{(2)}) \otimes c' = c \otimes g(c'_{(1)}) \otimes c'_{(2)}$, it follows that $g(c) \otimes c' = \varepsilon(c)g(c'_{(1)}) \otimes c'_{(2)}$, whence $c \otimes c' \mapsto \varepsilon(c)c'$ is left D -colinear.) For example, if $D = C$ and $g = \text{id}_C$, then $C \square_C C \cong C$, since $\underline{\Delta}$ is surjective.

It follows that C is in general isomorphic to a direct summand of $C \square_D C$ as D - C -bicomodules: ${}^D C \square_D C^C \cong {}^D C^C \oplus *$. Left codepth two coalgebra homomorphisms have the special complementary property:

Definition 6.1. A coalgebra homomorphism $g : C \rightarrow D$ is said to be *left codepth two* (*cD2*) if for some positive integer N ,

$$(14) \quad {}^D C \square_D C^C \oplus * \cong {}^D (C^N)^C,$$

i.e., the cotensor product $C \square_D C$ is isomorphic to a direct summand of a finite direct sum of C with itself as D - C -bicomodules. Right codepth two coalgebra homomorphisms are similarly defined.

The definition implies that there are D - C -bicomodule homomorphisms $f_i \in \text{Hom}^{D-C}(C, C \square_D C)$ and $g_i \in \text{Hom}^{D-C}(C \square_D C, C)$ such that

$$(15) \quad C \square_D C = \sum_{i=1}^N f_i \circ g_i.$$

We are then interested in obtaining simplications for these two hom-groups.

Proposition 6.2. *Given a coalgebra homomorphism $g : C \rightarrow D$,*

$$\text{End } {}^D C^D \cong \text{Hom}^{D-C}(C, C \square_D C).$$

Proof. Define a mapping $\text{End } {}^D C^D \rightarrow \text{Hom}^{D-C}(C, C \square_D C)$ by

$$(16) \quad \alpha \mapsto (\alpha \otimes C) \circ \underline{\Delta}.$$

Note that $\alpha(c_{(1)}) \otimes c_{(2)} \in C \square_D C$ for every $c \in C$, since $\alpha(c_{(1)}) \otimes g(c_{(2)}) = (C \otimes g)\Delta(\alpha(c))$ by right D -colinearity of α .

This mapping has inverse $\text{Hom}^{D-C}(C, C \square_D C) \rightarrow \text{End } {}^D C^D$ given by $F \mapsto (C \otimes \varepsilon) \circ F$. This is clearly left D -colinear, and right D -colinear since F and $C \otimes \varepsilon$ are so. It is an inverse since

$$(C \otimes \varepsilon) \circ (\alpha \otimes C) \circ \Delta = (\alpha \otimes \varepsilon) \circ \Delta = \alpha$$

for each $\alpha \in \text{End } {}^D C^D$, and right C -colinearity for $F \in \text{Hom}^{D-C}(C, C \square_D C)$ means $(C \otimes \Delta) \circ F = (F \otimes C) \circ \Delta$, so

$$(((C \otimes \varepsilon) \circ F) \otimes C) \circ \Delta = (C \otimes \varepsilon \otimes C) \circ (C \otimes \Delta) \circ F = F. \quad \square$$

Part of this proposition may be derived directly from the hom-cotensor relation [3, 11.10].

Let D^* and C^* be the dual K -algebras of coalgebras D and C , respectively, with multiplication given by the convolution product and unity element equal to the counit. Note that $g : C \rightarrow D$ induces the algebra homomorphism $g^* : D^* \rightarrow C^*$ given by $d^* \mapsto d^* \circ g$. Then C^* obtains a D^* - D^* -bimodule structure via g^* and we

define the centralizer $V_{C^*}(D^*)$ to be the set of all elements $c^* \in C^*$ such that for all $d^* \in D^*$, $c^*g^*(d^*) = g^*(d^*)c^*$ or equivalently suppressing g^* , $c^* \cdot d^* = d^* \cdot c^*$.

Lemma 6.3. *Given coalgebra homomorphism $g : C \rightarrow D$. Then $\text{End}^D C^C \cong V_{C^*}(D^*)$ via $f \mapsto \varepsilon \circ f$.*

Proof. If $f : C \rightarrow C$ is right C -colinear and left D -colinear, then for all $c \in C$, $f(c_{(1)}) \otimes c_{(2)} = \Delta(f(c))$ and $g(c_{(1)}) \otimes f(c_{(2)}) = (g \otimes C)\Delta(f(c))$, whence

$$g(c_{(1)})\varepsilon(f(c_{(2)})) = g(f(c)) = \varepsilon(f(c_{(1)}))g(c_{(2)}).$$

It follows that $g^*(d^*)(\varepsilon \circ f) = (\varepsilon \circ f)g^*(d^*)$ for all $d^* \in D^*$ w.r.t. the convolution product.

The inverse mapping $V_{C^*}(D^*) \rightarrow \text{End}^D C^C$ is given by

$$c^* \mapsto (c \mapsto c^*(c_{(1)})c_{(2)}).$$

We obtain $g(c_{(1)})c^*(c_{(2)}) = c^*(c_{(1)})g(c_{(2)})$ (for each $c \in C, c^* \in V_{C^*}(D^*)$) since there is equality when any $d^* \in D^*$ is applied (and any K -dual of coalgebras we consider “separates points”). It follows that the mapping $c \mapsto c^*(c_{(1)})c_{(2)}$ is left D -colinear. Of course, by right C -colinearity of f we have $\varepsilon(f(c_{(1)}))c_{(2)} = f(c)$ for each $c \in C$. It is easy to see that we have defined an anti-isomorphism of algebras. \square

Now suppose M is a D - C -bicomodule where $g : C \rightarrow D$ continues to be a coalgebra homomorphism. It is well-known that M then also has C^*-D^* -bimodule structure via convolution actions. From this we define the D^*-C^* -bimodule structure on the K -dual M^* by $(d^* \cdot m^* \cdot c^*)(m) = d^*(m_{(-1)})m^*(m_{(0)})c^*(m_{(1)})$. Similar to the lemma we prove:

Proposition 6.4. *If $g : C \rightarrow D$ is a coalgebra homomorphism and M is a D - C -bicomodule, then $\text{Hom}^{D-C}(M, C) \cong (M^*)^{D^*}$.*

Corollary 6.5. *Under the conditions above, we have $\text{Hom}^{D-C}(C \square_D C, C) \xrightarrow{\cong} (C \square_D C)^{D^*}$ via $f \mapsto \varepsilon \circ f$.*

Proof. We note that the inverse mapping is given by

$$(17) \quad \eta \mapsto (c \otimes c' \mapsto \eta(c \otimes c'_{(1)})c'_{(2)}) \quad \square$$

6.1. Left coD2 quasibases. We are now in a position to re-write eq. (15) identifying $f_i \in \text{Hom}^{D-C}(C, C \square_D C)$ with $\alpha_i \in \text{End}^D C^D$ using the mapping (16), and identifying $g_i \in \text{Hom}^{D-C}(C \square_D C, C)$ with $\eta_i \in (C \square_D C)^{D^*}$ using mapping (17). We obtain for each $c \otimes c' \in C \square_D C$:

$$(18) \quad c \otimes c' = \sum_{i=1}^N \eta_i(c \otimes c'_{(1)})\alpha_i(c'_{(2)}) \otimes c'_{(3)}.$$

The equation is analogous to the eq. (2); for that reason we call $\eta_i \in (C \square_D C)^{D^*}$ and $\alpha_i \in \text{End}^D C^D$ *left coD2 quasibases* for the coalgebra homomorphism $g : C \rightarrow D$. The quasibase equation above has the equivalent form,

$$(19) \quad C \square_D C = \sum_{i=1}^N (\eta_i \otimes \alpha_i \otimes C) \circ (C \otimes \Delta^2)$$

Given $\beta \in \text{End } {}^D C^D$, we note that for $c \in C$ we have $\beta(c_{(1)}) \otimes c_{(2)} \in C \square_D C$, and substituting this into eq. (18) and applying $C \otimes \varepsilon$ to this yields

$$(20) \quad \beta(c) = \sum_{i=1}^N \eta_i(\beta(c_{(1)}) \otimes c_{(2)}) \alpha_i(c_{(3)}),$$

in other words, $\beta = \sum_i (\eta_i \otimes \alpha_i) \circ (\beta \otimes C \otimes C) \circ \Delta^2$.

6.2. Right bialgebroid structure on $\text{End } {}^D C^D$ over C^{*D^*} . We proceed to show that a coD2 coalgebra homomorphism $g : C \rightarrow D$ has bialgebroid structure on $\text{End } {}^D C^D$. This and its noncommutative base algebra will be denoted by

$$(21) \quad E := \text{End } {}^D C^D, \quad R := C^{*D^*}.$$

There are immediately two commuting mappings, a homomorphism $s : R \rightarrow E$ and an anti-homomorphism $t : R \rightarrow E$, given by ($r \in R, c \in C$)

$$(22) \quad s(r)(c) = c_{(1)} r(c_{(2)}),$$

a source map, and a target map,

$$(23) \quad t(r)(c) = r(c_{(1)}) c_{(2)}.$$

We note that $s(r)t(r') = t(r')s(r)$ w.r.t. composition in E since both applied to $c \in C$ yield $r'(c_{(1)})r(c_{(3)})c_{(2)}$ by coassociativity of Δ . We note that the R - R -bimodule structure induced by s and t from the right, suggestively denoted by $E_{s,t}$ is given by the straightforward

$$(24) \quad (r \cdot \alpha \cdot r')(c) = r(c_{(1)}) \alpha(c_{(2)}) r'(c_{(3)})$$

for $r, r' \in R, \alpha \in E, c \in C$.

At this point we may profitably note that eq. (20) shows that ${}_R E$ is f.g. projective, since the N mappings $\beta \mapsto \eta_i \circ (\beta \otimes C) \circ \Delta$ are easily seen to be in $\text{Hom}({}_R E, {}_R R)$.

The counit map $\varepsilon_E : E \rightarrow R$ is given by

$$(25) \quad \varepsilon_E(\alpha) = \varepsilon \circ \alpha$$

where $\varepsilon : C \rightarrow K$ denotes the counit on C . Since α in E is right and left D -colinear, it follows that $g(c_{(1)})(\varepsilon \circ \alpha)(c_{(2)}) = (\varepsilon \circ \alpha)(c_{(1)})g(c_{(2)})$ for each $c \in C$ since both equal $g(\alpha(c))$. Then of course $\varepsilon \circ \alpha \in C^{*D^*} = R$. Note that $\varepsilon_E(\text{id}_C) = \varepsilon = 1_R$. Note the ε_E is left and right R -linear:

$$\varepsilon_E(r \cdot \alpha \cdot r')(c) = r(c_{(1)}) \varepsilon(\alpha(c_{(2)})) r'(c_{(3)}) = (r \varepsilon_E(\alpha) r')(c)$$

w.r.t. the convolution product in R . Also note that $(\alpha, \beta \in E, c \in C)$

$$\begin{aligned} \varepsilon_E(s(\varepsilon_E(\alpha)) \circ \beta)(c) &= \varepsilon \circ s(\varepsilon_E(\alpha))(\beta(c)) \\ &= \varepsilon(\beta(c)_{(1)}) \varepsilon(\alpha(\beta(c)_{(2)})) \\ &= \varepsilon_E(\alpha \circ \beta)(c) \end{aligned}$$

Thus, $\varepsilon_E(\alpha \circ \beta) = \varepsilon_E(s(\varepsilon_E(\alpha)) \circ \beta)$ and similarly we compute $\varepsilon_E(\alpha \circ \beta) = \varepsilon_E(t(\varepsilon_E(\alpha)) \circ \beta)$.

Dualizing the definition of generalized Lu coproduct in [12, eq. (74)], we would obtain a coproduct on E given by $\Delta_E(\alpha) = \Delta \circ \alpha$ in $\text{Hom}^{D-D}(C, C \square_D C)$. In order to make sense of this, we need:

Proposition 6.6. *If $g : C \rightarrow D$ is left coD2, then there is an isomorphism,*

$$(26) \quad E \otimes_R E \xrightarrow{\cong} \text{Hom}^{D-D}(C, C \square_D C), \quad \alpha \otimes \beta \mapsto (c \mapsto \alpha(c_{(1)}) \otimes \beta(c_{(2)}))$$

Proof. It is clear from D -colinearity of α and β as well as eq. (24) that the mapping $\alpha \otimes_R \beta \mapsto (\alpha \otimes \beta) \circ \underline{\Delta}$ is well-defined in all respects.

An inverse mapping $\text{Hom}^{D-D}(C, C \square_D C) \rightarrow E \otimes_R E$ is given in terms of the left coD2 quasibases $\eta_i \in (C \square_D C)^{*D^*}$ and $\alpha_i \in E$ in eq. (18):

$$(27) \quad G \mapsto \sum_{i=1}^N (C \otimes \eta_i) \circ (G \otimes C) \circ \underline{\Delta} \otimes_R \alpha_i,$$

where $G \in \text{Hom}^{D-D}(C, C \square_D C)$. It is easy to see that $(C \otimes \eta_i) \circ (G \otimes C) \circ \underline{\Delta}$ is left D -colinear, but it is right D -colinear as well, since $(c \in C)$

$$(C \otimes g) \Delta(C \otimes \eta_i)(G(c_{(1)}) \otimes c_{(2)}) = (C \otimes \eta_i)(G(c_{(1)}) \otimes c_{(2)}) \otimes g(c_{(3)})$$

follows from G taking values in $C \square_D C$ and η_i satisfying $g(c_{(1)})\eta_i(c_{(2)} \otimes c') = \eta_i(c \otimes c'_{(1)})g(c'_{(2)})$ for $c \otimes c' \in C \square_D C$. By eqs. (20) and (24) this mapping sends

$$(\alpha \otimes \beta) \circ \underline{\Delta} \mapsto \sum_{i=1}^N \alpha \otimes_R (\eta_i \circ (\beta \otimes C) \circ \underline{\Delta}) \cdot \alpha_i = \alpha \otimes \beta,$$

where $\eta_i \circ (\beta \otimes C) \circ \underline{\Delta} \in R$.

Conversely, note that the image of $G \in \text{Hom}^{D-D}(C, C \square_D C)$ in $E \otimes_R E$ is sent into

$$\sum_{i=1}^N (C \otimes \eta_i \otimes C) \circ (G \otimes C \otimes \alpha_i) \circ \underline{\Delta}^2 = G,$$

since $\text{id}_{C \square_D C} = \sum_{i=1}^N (\eta_i \otimes \alpha_i \otimes C) \circ (C \otimes \underline{\Delta}^2)$. \square

Note that C has left R -module structure that commutes with the right coaction $\rho^R : C \rightarrow C \otimes D$. The left module action is naturally $r \cdot c = c_{(1)}r(c_{(2)})$ where $r \in R = C^{*D^*}$. Then note that

$$\rho^R(r \cdot c) = c_{(1)} \otimes g(c_{(2)})r(c_{(3)}) = c_{(1)}r(c_{(2)}) \otimes g(c_{(3)}) = r \cdot \rho^R(c).$$

Similarly, ${}^D C_R$ has associative action and coaction. It follows that $\text{Hom}^{D-D}(C, C \square_D C)$ has R - R -bimodule structure induced by its contravariant argument:

$$(28) \quad (r \cdot G \cdot r')(c) = r(c_{(1)})G(c_{(2)})r'(c_{(3)}) \quad (r \in R, c \in C, G \in \text{Hom}^{D-D}(C, C \square_D C))$$

Also $E \otimes_R E$ has a natural R - R -bimodule structure derived from ${}_R E$ and E_R in the first and second tensorands. The isomorphism in the proposition is clearly left and right R -linear:

Corollary 6.7. *The isomorphism $E \otimes_R E \xrightarrow{\cong} \text{Hom}^{D-D}(C, C \square_D C)$, given by $\alpha \otimes_R \beta \mapsto (\alpha \otimes \beta) \circ \underline{\Delta}$, is an R - R -bimodule isomorphism.*

By identifying $E \otimes_R E$ with $\text{Hom}^{D-D}(C, C \square_D C)$, we define a comultiplication for E by

$$(29) \quad \Delta_E(\alpha) = \underline{\Delta} \circ \alpha,$$

clearly a right and left D -colinear homomorphism from C into $C \square_D C$ for each $\alpha \in E = \text{End } {}^D C^D$. We note that Δ_E is R - R -linear:

$$\Delta_E(r \cdot \alpha \cdot r')(c) = r(c_{(1)}) \underline{\Delta}(\alpha(c_{(2)})) r'(c_{(3)}) = (r \cdot \Delta_E(\alpha) \cdot r')(c),$$

by eq. (28).

Note that $\Delta_E(1_E) = 1_E \otimes_R 1_E$, since $\Delta_E(\text{id}_C) = \underline{\Delta}$, which corresponds to $\text{id}_C \otimes \text{id}_C$ under the identification mapping (26).

Next we show that $(\varepsilon_E \otimes_R E) \circ \Delta_E = E$ (where E denotes id_E). Note that via the identification mapping (27) the formal definition of the comultiplication $\Delta_E : E \rightarrow E \otimes_R E$ is $(\beta \in E)$

$$(30) \quad \Delta_E(\beta) = \sum_{i=1}^N (C \otimes \eta_i) \circ ((\underline{\Delta} \circ \beta) \otimes C) \circ \underline{\Delta} \otimes \alpha_i.$$

Whence identifying $R \otimes_R E \cong E$ canonically and letting $c \in C$:

$$\begin{aligned} (\varepsilon_E \otimes_R E) \circ \Delta_E(\beta)(c) &= \sum_{i=1}^N \varepsilon(\beta(c_{(1)})_{(1)}) \eta_i(\beta(c_{(1)})_{(2)} \otimes c_{(2)}) \alpha_i(c_{(3)}) \\ &= \sum_i \eta_i(\beta(c_{(1)}) \otimes c_{(2)}) \alpha_i(c_{(3)}) = \beta(c), \end{aligned}$$

by eq. (20).

Next note that $(E \otimes_R \varepsilon_E) \circ \Delta_E = E$ as follows. Apply $\Delta \otimes C$ to the quasibases eq. (18) for $c \otimes c' \in C \square_D C$ to obtain

$$c_{(1)} \otimes c_{(2)} \otimes c' = \sum_{i=1}^N c_{(1)} \eta_i(c_{(2)} \otimes c'_{(1)}) \otimes \alpha_i(c'_{(2)}) \otimes c'_{(3)},$$

to which we apply $C \otimes \varepsilon \otimes \varepsilon$, obtaining

$$(31) \quad c\varepsilon(c') = \sum_i c_{(1)} \eta_i(c_{(2)} \otimes c'_{(1)}) \varepsilon(\alpha_i(c'_{(2)})).$$

We apply this to $\beta(c_{(1)}) \otimes c_{(2)} \in C \square_D C$ to see that

$$\begin{aligned} ((E \otimes \varepsilon_E) \circ \Delta_E)(\beta)(c) &= \sum_i \beta(c_{(1)})_{(1)} \eta_i(\beta(c_{(1)})_{(2)} \otimes c_{(2)}) \varepsilon(\alpha_i(c_{(3)})) \\ &= \beta(c_{(1)}) \varepsilon(c_{(2)}) = \beta(c). \end{aligned}$$

Denote the values $\Delta_E(\beta) = \beta_{(1)} \otimes_R \beta_{(2)} \in E \otimes_R E$ using Sweedler's notation. In $\text{Hom}^{D-D}(C, C \square_D C)$ this is identified with the mapping $c \mapsto \beta_{(1)}(c_{(1)}) \otimes \beta_{(2)}(c_{(2)})$. At the same time, we define $\Delta_E(\beta) = \underline{\Delta} \circ \beta$ in this same hom-group, thus for every $\beta \in E$ and $c \in C$,

$$(32) \quad \beta(c)_{(1)} \otimes \beta(c)_{(2)} = \beta_{(1)}(c_{(1)}) \otimes \beta_{(2)}(c_{(2)}),$$

as an equality in $C \otimes_K C$.

Now note that for each $r \in R$ and $\alpha \in E$, we have

$$\begin{aligned} (s(r) \circ \alpha_{(1)})(c_{(1)}) \otimes \alpha_{(2)}(c_{(2)}) &= s(r)(\alpha(c)_{(1)}) \otimes \alpha(c)_{(2)} \\ &= \alpha(c)_{(1)} r(\alpha(c)_{(2)}) \otimes \alpha(c)_{(3)} \\ &= \alpha(c)_{(1)} \otimes t(r)(\alpha(c)_{(2)}) \\ &= \alpha_{(1)}(c_{(1)}) \otimes (t(r) \circ \alpha_{(2)})(c_{(2)}), \end{aligned}$$

whence $\text{Im } \Delta_E \subseteq (E \otimes_R E)^R$ where the R - R -bimodule structure on $E \otimes_R E$ uses ${}_{s,t}E$ instead.

As is well-known in the theory of bialgebroids, the last condition on $\text{Im } \Delta_E$ shows that there is a well-defined tensor algebra multiplication on $\text{Im } \Delta_E$. We claim that

$$(33) \quad \Delta_E(\alpha \circ \beta) = \alpha_{(1)} \circ \beta_{(1)} \otimes_R \alpha_{(2)} \circ \beta_{(2)}$$

Using the identification $E \otimes_R E \cong \text{Hom}^{D-D}(C, C \square_D C)$ in proposition 6.6 again, for $c \in C$,

$$\begin{aligned} \alpha_{(1)}(\beta_{(1)}(c_{(1)})) \otimes \alpha_{(2)}(\beta_{(2)}(c_{(2)})) &= \alpha_{(1)}(\beta(c)_{(1)}) \otimes \alpha_{(2)}(\beta(c)_{(2)}) \\ \alpha(\beta(c))_{(1)} \otimes \alpha(\beta(c))_{(2)} &= (\alpha \circ \beta)_{(1)}(c_{(1)}) \otimes (\alpha \circ \beta)_{(2)}(c_{(2)}), \end{aligned}$$

hence $\Delta_E(\alpha \circ \beta) = \Delta_E(\alpha)\Delta_E(\beta)$ in this special submodule of $E \otimes_R E$ where tensor algebra multiplication is valid.

Finally we note that Δ_E is coassociative. Heuristically this depends on the coassociativity of the coproduct Δ on C since $\Delta_E(\alpha) = \underline{\Delta} \circ \alpha$ where $\underline{\Delta}$ is just Δ with a restriction of its codomain. To prove coassociativity we need a lemma generalizing proposition 6.6:

Lemma 6.8. *Suppose M is a C - C -bicomodule and $g : C \rightarrow D$ a coalgebra homomorphism of codepth two. Then there is a natural R - R -bimodule isomorphism,*

$$\text{Hom}^{D-D}(C, M) \otimes_R E \cong \text{Hom}^{D-D}(C, M \square_D C),$$

via $f \otimes_R \alpha \mapsto (c \mapsto f(c_{(1)}) \otimes \alpha(c_{(2)}))$.

Proof. The proof is almost identical to the proof of proposition 6.6, and is therefore omitted. \square

Applying the lemma with $M = C \square_D C$, proposition 6.6 and using coassociativity of cotensor product [3, 11.6], we note that

$$(34) \quad E \otimes_R E \otimes_R E \cong \text{Hom}^{D-D}(C, C \square_D C \square_D C)$$

via

$$\alpha \otimes_R \beta \otimes_R \gamma \mapsto (c \mapsto \alpha(c_{(1)}) \otimes \beta(c_{(2)}) \otimes \gamma(c_{(3)})).$$

It follows from eq. (32) that $(\Delta_E \otimes E)\Delta_E(\alpha)$ is identified with

$$\alpha_{(1)}(c_{(1)})_{(1)} \otimes \alpha_{(1)}(c_{(1)})_{(2)} \otimes \alpha_{(2)}(c_{(2)}) = \alpha_{(1,1)}(c_{(1)}) \otimes \alpha_{(1,2)}(c_{(2)}) \otimes \alpha_{(2)}(c_{(3)}),$$

but the LHS equals

$$\alpha_{(1)}(c_{(1)}) \otimes \alpha_{(2)}(c_{(2)})_{(1)} \otimes \alpha_{(2)}(c_{(2)})_{(2)} = \alpha_{(1)}(c_{(1)}) \otimes \alpha_{(2,1)}(c_{(2)}) \otimes \alpha_{(2,2)}(c_{(3)})$$

which is identified with $(E \otimes_R \Delta_E)\Delta_E(\alpha)$. Via the isomorphism (34) we obtain $(\Delta_E \otimes_R E) \circ \Delta_E = (E \otimes_R \Delta_E) \circ \Delta_E$.

We have proven:

Theorem 6.9. *If $g : C \rightarrow D$ is a left codepth two homomorphism of coalgebras, then $E = \text{End}^D C^D$ is a right bialgebroid over $R = C^{*D*}$, the centralizer subalgebra in C^* induced by the dual algebra homomorphism $g^* : D^* \rightarrow C^*$. Moreover, the left*

R-module E is finitely generated projective. The structure of the *R*-bialgebroid and endomorphism algebra E is given by $(r, r' \in R, c \in C, \alpha \in E)$

$$(35) \quad s(r)(c) = c_{(1)}r(c_{(2)})$$

$$(36) \quad t(r)(c) = r(c_{(1)})c_{(2)}$$

$$(37) \quad (r \cdot \alpha \cdot r')(c) = r(c_{(1)})\alpha(c_{(2)})r'(c_{(3)})$$

$$(38) \quad \Delta_E(\alpha) = \sum_{i=1}^N (C \otimes \eta_i) \circ ((\underline{\Delta} \circ \alpha) \otimes C) \circ \underline{\Delta} \otimes_R \alpha_i$$

$$(39) \quad \varepsilon_E(\alpha) = \varepsilon \circ \alpha$$

with respect to the codepth two quasibases $\alpha_i \in E$ and $\eta_i \in (C \square_D C)^{*D^*}$ satisfying for all $c \otimes c' \in C \square_D C$,

$$(40) \quad c \otimes c' = \sum_{i=1}^N \eta_i(c \otimes c'_{(1)})\alpha_i(c'_{(2)}) \otimes c'_{(3)}.$$

Proof. For the convenience of the reader, we gather the axioms of a right *R*-bialgebroid E , verified in the subsection preceding the theorem. Given *K*-algebras *R* and *E*, a bialgebroid E over *R* has “source” algebra homomorphism $s : R \rightarrow E$ and “target” algebra anti-homomorphism $t : R \rightarrow E$ such that commutativity $s(r)t(r') = t(r')s(r)$ holds within E for all $r, r' \in R$. We then refer to an *R*-*R*-bimodule structure on E induced by $r \cdot e \cdot r' = es(r')t(r)$ for all $r, r' \in R, e \in E$, and the natural *R*-*R*-bimodule structures on R and $E \otimes_R E$. The axioms are then:

- (1) There is an *R*-coring $(E, R, \Delta_E, \varepsilon_E)$: i.e., “comultiplication” $\Delta_E : E \rightarrow E \otimes_R E$ and “counit” $\varepsilon_E : E \rightarrow R$ are right and left *R*-linear, Δ_E is coassociative, and ε_E satisfies counitality axioms;
- (2) The comultiplication and counit are unit-preserving: $\Delta_E(1_E) = 1_E \otimes_R 1_E$ and $\varepsilon_E(1_E) = 1_R$;
- (3) the comultiplication takes its values in the submodule of finite sums,

$$E \times_R E := \left\{ \sum_i x_i \otimes_R y_i \in E \otimes_R E \mid \forall r \in R, \sum_i s(r)x_i \otimes_R y_i = \sum_i x_i \otimes_R t(r)y_i \right\}$$

where tensor algebra multiplication is well-defined;

- (4) for all $e, e' \in E$, we have $\Delta_E(ee') = \Delta_E(e)\Delta_E(e')$;
- (5) the unital tensor category axiom for *R* as an E -module:

$$\varepsilon_E(ee') = \varepsilon_E(s(\varepsilon_E(e))e') = \varepsilon_E(t(\varepsilon_E(e))e')$$

for all $e, e' \in E$. □

6.3. Discussion. A theory of right coD2 coalgebra homomorphisms has a similar development. By consulting [3], one might under suitable hypotheses develop a similar theory of codepth two for a homomorphism of *R*-corings. The (“co-Sweedler”) *C*-ring $C \square_D C$ for any coalgebra homomorphism $g : C \rightarrow D$ is defined by $\mu : C \square_D C \square_D C \rightarrow C \square_D C$, $\mu(c \otimes c' \otimes c'') = c\varepsilon(c') \otimes c''$ with unit $\eta = \underline{\Delta} : C \rightarrow C \square_D C$. Something related to this *C*-ring might play a role in a more complete theory of codepth two.

If the coalgebra homomorphism $g : C \rightarrow D$ is coD2 and its dual $g^* : D^* \rightarrow C^*$ is D2, then there is an anti-monomorphism of *R*-bialgebroids $E \rightarrow S := \text{End}_{D^*} C_{D^*}^*$ given by $\alpha \mapsto \hat{\alpha}$, where $\hat{\alpha}(c^*) = c^* \circ \alpha$. If *C* and *D* are finite dimensional, this is an anti-isomorphism of a left and right bialgebroid over *R*. It would be interesting

to know something of the precise relationship between D2 and coD2. For example, the quotient homomorphism $H \rightarrow \overline{H}$ should be coD2 if $K \hookrightarrow H$ is a normal Hopf subalgebra with $\overline{H} = H/HK^+$.

Given a coD2 coalgebra homomorphism $g : C \rightarrow D$ and its constructions E and R defined above, the coalgebra C is a left E -module coalgebra (i.e. a coalgebra in the tensor category of left E -modules) due to eq. (32) and one other, counital axiom. If ${}^D C$ is co-balanced, it should be that $\ker g = \{\alpha(c) - c_{(1)}\varepsilon(\alpha(c_{(2)})) | c \in C, \alpha \in E\}$ and we obtain the image of g in D in a type of coGalois theory.

It would be interesting to pursue the rest of the structure of duality, e.g. endomorphism algebras and smash products, realization of the R -dual left bialgebroid of E , and its relationship to [4].

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